Atiyah classes of Lie algebroid homotopy modules

Panagiotis Batakidis, Aristotle University of Thessaloniki

Sylvain Lavau, Ruđer Bošković Institute, Zagreb

Introduction

We consider Lie algebroid pairs $A \subset L$ and provide here a definition of the Atiyah class of the extension of a representation up to homotopy E of A to an L-superconnection on E. We show that the construction is consistent with the traditional notion of Atiyah class of Lie algebroid pairs when the representation up to homotopy E is (a resolution of) a classical Lie algebroid representation. Namely, in that particular case, the former does not contain any further information that is not already contained in the latter. arXiv:2406.05204

Reminder about Atiyah classes in the classical case

Let $A \subset L$ be two Lie algebroids over M, and let K be a *Lie algebroid representation of* A, i.e. there is a flat A-connection ∇ on K. Extend this connection to a L-connection on K. It is non-necessarily flat, so in particular the one-form $\mathrm{at}_K \in \Omega^1(A, A^\circ \otimes \mathrm{End}(K))$ defined by:

$$\mathrm{at}_K(a; l) = R_\nabla(a, \ell) = [\nabla_a, \nabla_\ell] - \nabla_{[a,\ell]},$$

for any $l \in \Gamma(L/A)$ with preimage $\ell \in \Gamma(L)$, is not necessarily zero. This one-form is closed, and it is called the *Atiyah cocycle* of K w.r.t. the Lie pair $A \subset L$. Its cohomology class does not depend on the extension of ∇ to L, and it vanishes iff there exists an A-compatible L-connection on K.

Atiyah classes of homotopy A-modules

Now let $E = (E_i)_{i\in\mathbb{Z}}$ be a (split, finite dimensional) graded vector bundle over M. We call *A-superconnection* any differential operator $D_A : \Omega(A, E)_\blacktriangle \to \Omega(A, E)_{\blacktriangle+1}$ of total degree $+1$ splitting into a sum

$$D_A = \partial + d_A^\nabla + \sum_{k\geq 2}^{\mathrm{rk}(A)} \omega_A^{(k)} \wedge \cdot.$$

where $\partial : E_\bullet \to E_{\bullet+1}$ is a vector bundle morphism, ∇ is a A-connection on E, and $\omega_A^{(k)} \in \Omega^k(A, \mathrm{End}(E)_{1-k})$ are *connection k-forms*. The graded vector bundle E is said to be a *homotopy A-module* whenever $(D_A)^2 = 0$, i.e. when:

$$[d_A^\nabla, \partial] = 0 \quad \text{and} \quad R_A^{(k)} + \big[\partial, \omega_A^{(k)}\big] = 0 \quad \forall\, k \geq 2,$$

where $R_A^{(k)}$ is the *curvature k-form* associated to the connection $k-1$-form:

$$R_A^{(k)} = d_A\omega_A^{(k-1)} + \sum_{\substack{1\leq s,t\leq k-1\\ s+t=k}} \omega_A^{(s)} \wedge \omega_A^{(t)}.$$

Assume that D_A is extended to a L-superconnection D_L on E. It does not necessarily satisfy $(D_L)^2 = 0$. Any $p+1$-form $\omega \in \Omega^{p+1}(L, \mathrm{End}(E))$ such that $\omega|_{\wedge^{p+1}A} = 0$ induces a p-form $\varpi \in \Omega^p(A, A^\circ \otimes \mathrm{End}(E))$ whenever restricted to $\wedge^p A \otimes L/A$. Namely, if one denotes $\ell \in \Gamma(L)$ a preimage of $l \in \Gamma(L/A)$, then:

$$\varpi(a_1, \ldots, a_p; l) = \omega(a_1, \ldots, a_p, \ell). \tag{1}$$

Define a differential operator s on $\widehat{\Omega}(E)^{\bullet,\bullet} = \Omega^\bullet(A, A^\circ \otimes \mathrm{End}_\bullet(E))$ by

$$s^{(k)}(\varpi)(a_1, \ldots, a_{k+p}; l) = [D_L^{(k)}, \omega](a_1, \ldots, a_{k+p}, \ell). \tag{2}$$

Proposition 0.1. *Equation (2) does not depend on the choice of L-superconnection D_L extending the A-superconnection D_A, and the A-superconnection s on $A^\circ \otimes \mathrm{End}(E)$ defines a homotopy A-module structure, i.e. it is a total degree $+1$ differential on the bigraded vector space $\widehat{\Omega}(E)^{\bullet,\bullet}$.*

Then, we set

$$a^{(1)} = [d_L^\nabla, \partial] \quad \text{and} \quad a^{(k)} = R_L^{(k)} + [\partial, \omega_L^{(k)}] \quad \forall\, k \geq 2.$$

By construction, for every $k \geq 1$, we have that $a^{(k)}\big|_{\wedge^k A} = 0$. We deduce that the $k+1$-forms $a^{(k+1)} \in \Omega^{k+1}(L, \mathrm{End}_{1-k}(E))$ give rise, through Equation (1), to a family of k-forms $\alpha^{(k)} \in \Omega^k(A, A^\circ \otimes \mathrm{End}_{1-k}(E))$ of total degree $+1$ in $\widehat{\Omega}(E)^{\bullet,\bullet}$:

$$\alpha^{(0)}(l) = [\nabla_\ell, \partial],$$

$$\alpha^{(k)}(a_1, \ldots, a_k; l) = R_L^{(k+1)}(a_1, \ldots, a_k, \ell) + [\partial, \omega_L^{(k+1)}](a_1, \ldots, a_k, \ell).$$

Theorem. *We have the following statement generalizing the classical case:*

1. *The form $\alpha = \sum_k \alpha^{(k)}$ is a cocycle of the cochain complex $\big(\widehat{\Omega}(E), s\big)$;*
2. *Its cohomology class $[\alpha] \in H^1\big(\widehat{\Omega}(E), s\big)$ does not depend on the choice of extension D_L of the A-superconnection D_A;*
3. *$[\alpha] = 0$ iff there exists a homotopy A-compatible L-superconnection on E.*

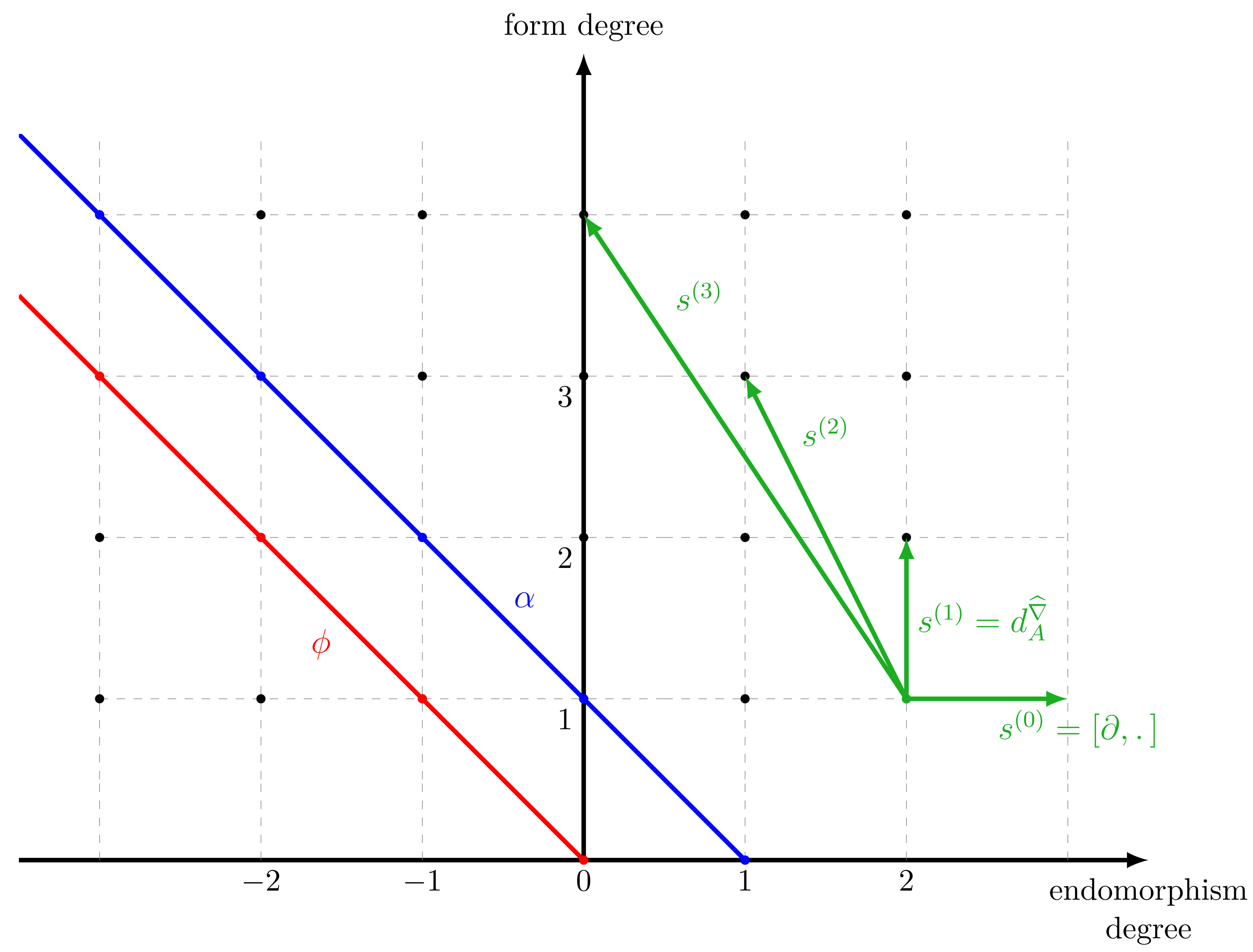

Figure 1: Representation of the bi-graded space $\widehat{\Omega}(E)^{\bullet,\bullet} = \Omega^\bullet(A, A^\circ \otimes \mathrm{End}_\bullet(E))$ and an Atiyah cocycle α of total degree $+1$, together with an element ϕ of total degree 0 and the total degree $+1$ differential. The component $\alpha^{(p)}$ sits at the node of coordinates $(1-p, p)$.

Relationship with Atiyah classes in the classical case

Assume that the homotopy A-module E is only graded over non-positive degrees, i.e $E_\bullet = \bigoplus_{-n\leq i\leq 0} E_i$ for some $n \geq 0$. Assume moreover that the chain complex (E, ∂) is *regular*, i.e. that for every $i \leq 0$, the vector bundle morphism $\partial : E_i \to E_{i+1}$ has constant rank. This allows to define the cohomology of the chain complex (E, ∂) as the following quotient vector bundle:

$$\mathcal{H}^i(E, \partial) = \mathrm{Ker}(\partial : E_i \to E_{i+1}) \Big/ \mathrm{Im}(\partial : E_{i-1} \to E_i).$$

When the cohomology $\mathcal{H}^\bullet(E, \partial)$ is concentrated in degree 0, we set $K = \mathcal{H}^0(E, \partial)$, so that the chain complex (E, ∂) is a resolution of the vector bundle K.

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\partial} & E_{-2} & \xrightarrow{\partial} & E_{-1} & \xrightarrow{\partial} & E_0 & \xrightarrow{0} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{0} & 0
\end{array}$$

Theorem. *For every $p \in \mathbb{Z}$,*

$$H^p\big(\widehat{\Omega}(E), s\big) \simeq H^p\big(A, A^\circ \otimes \mathrm{End}(K)\big). \tag{3}$$

Moreover, the Atiyah class $[\alpha_E] \in \widehat{H}^1(E, s)$ of the homotopy A-module E is sent by the isomorphism (3) to the Atiyah class $[\mathrm{at}_K] \in H^1\big(A, A^\circ \otimes \mathrm{End}(K)\big)$ of the Lie algebroid representation K, so we have that $[\alpha_E] = 0$ iff $[\mathrm{at}_K] = 0$.

This theorem hence establishes that the components $\alpha_E^{(k)}$ (for $k \neq 1$) do not contain more cohomological information that is not already contained in at_K.